

THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS FROM $\mathcal{C}(K)$ TO UNIFORMLY CONVEX SPACES

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ABSTRACT. We show that the pair $(\mathcal{C}(K), X)$ has the Bishop-Phelps-Bollobás property for operators if K is a compact Hausdorff space and X is a uniformly convex space.

1. INTRODUCTION

In this paper, we deal with strengthening of the famous Bishop-Phelps theorem. In 1961, Bishop and Phelps [8] showed that the set of all norm attaining functionals on a Banach space X is dense in its dual space X^* which is now called Bishop-Phelps theorem. This theorem has been extended to operators between Banach spaces X and Y . In general, the set of norm attaining operators $\mathcal{NA}(X, Y)$ is not dense in the space of linear operators $\mathcal{L}(X, Y)$. However, it is true for some pair of Banach spaces (X, Y) . One of very well-known examples is the pair of every reflexive Banach space X and every Banach space Y , which was shown by Lindenstrauss [24]. After that, this was generalized by Bourgain to Banach space X with Radon-Nikodým property [10], and also there have been many efforts to find other positive examples [12, 13, 15, 17, 19, 26, 27].

Meanwhile, Bollobás sharpened Bishop-Phelps theorem as follows. From now on, the unit ball and the unit sphere of a Banach space X will be denoted by B_X and S_X , respectively.

Theorem 1.1. ([9]) *For an arbitrary $\epsilon > 0$, if $x^* \in S_{X^*}$ satisfies $|1 - x^*(x)| < \frac{\epsilon^2}{4}$ for $x \in B_X$, then there are both $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.*

This Bishop-Phelps-Bollobás theorem shows that if a functional almost attains its norm at a point, then it is possible to approximate simultaneously both the functional and the point by norm attaining functionals and their corresponding norm attaining points. Clearly, Bishop-Phelps-Bollobás theorem implies Bishop-Phelps theorem.

Similarly to the case of Bishop-Phelps theorem, Acosta, Aron, García and Maestre [1] started to extend this theorem to bounded linear operators between Banach spaces and introduced the new notion *Bishop-Phelps-Bollobás property*.

Definition 1.2. ([1, Definition 1.1]) Let X and Y be Banach spaces. We say that the pair (X, Y) has the Bishop-Phelps-Bollobás property for operators (*BPBp*) if, given $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that if there exist both $T \in S_{\mathcal{L}(X, Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist both an operator $S \in S_{\mathcal{L}(X, Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|T - S\| < \epsilon.$$

Acosta et al. showed [1] that the pair (X, Y) has the *BPBp* for finite dimensional Banach spaces X and Y , and that the pair (ℓ_∞^n, Y) has the *BPBp* for every n if Y is a uniformly convex space. In the same paper, they asked if the pairs (c_0, Y) and (ℓ_∞, Y) have the *BPBp* for uniformly convex spaces Y . The

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first author solved the c_0 case and proved [20] that (c_0, Y) have the Bishop-Phelps-Bollobás property for all uniformly convex spaces Y .

Let $X = L_\infty(\mu)$ or $X = c_0(\Gamma)$ for a set Γ . Very recently, Lin and authors [23] proved that (X, Y) has the BPBp for every uniformly convex space Y . So $(L_\infty(\mu), L_p(\nu))$ has the BPBp for all $1 < p < \infty$ and for all measures ν . They also proved that (X, Y) , as a pair of complex spaces, has the BPBp for every uniformly complex convex space Y . In particular, $(L_\infty(\mu), L_1(\nu))$, as a pair of complex spaces, has the BPBp, since $L_1(\nu)$ is uniformly complex convex [18].

On the other hand, there have been several researches about the BPBp for operators into $C(K)$ spaces (or uniform algebras). Even though Schachermayer showed [26] that the set of norm attaining operators is not dense in $\mathcal{L}(L_1[0, 1], C[0, 1])$, there are some positive results about the BPBp. It is shown [4] that $(X, C(K))$ has the BPBp if X is an Asplund space. This result was extended so that (X, A) has the BPBp if X is Asplund and A is a uniform algebra [11]. The authors also proved [21] that $(X, C(K))$ has the BPBp if X^* admits a uniformly simultaneously continuous retractions. It is also worthwhile to remark that the pair $(C(K), C(L))$ of the spaces of real-valued continuous functions has the BPBp for every compact Hausdorff spaces K and L [2]. Concerning the results about L_∞ spaces, it is shown [7] that $(L_1(\mu), L_\infty[0, 1])$ has the BPBp and this was generalized [14] so that $(L_1(\mu), L_\infty(\nu))$ has the BPBp if μ is any measure and ν is a localizable measure. These are the strengthening of the results that the set of norm-attaining operators is dense in $\mathcal{L}(L_1(\mu), L_\infty(\nu))$ [17, 25] for every measure μ and every localizable measure ν . Finally we remark that if X is uniformly convex, then (X, Y) has the BPBp for every Banach space Y [3, 5, 22].

Throughout this paper, we consider only real Banach spaces. It is the main result of this paper that $(C(K), X)$ has the BPBp for every compact Hausdorff space K and for every uniformly convex space X . Recall that Schachermayer showed [26] that every weakly compact operator from $C(K)$ into a Banach space can be approximated by norm attaining weakly compact operators (cf. [6, Theorem 2]). So the set of all norm attaining operators is dense in $\mathcal{L}(C(K), Y)$ for every reflexive space Y . Notice that the reflexivity of Y is not sufficient to prove that $(C(K), Y)$ has the BPBp. Indeed, if we take a reflexive strictly convex space Y_0 which is not uniformly convex, then $(\ell_1^{(2)}, Y_0)$ does not have the BPBp [1, 5]. If we take K_0 as the set consisting of only two points, then $C(K_0)$ is isometrically isomorphic to 2-dimensional $\ell_1^{(2)}$ space. Hence $(C(K_0), Y_0)$ does not have the BPBp. However, if X is uniformly convex, then it will be shown that $(C(K), X)$ has the BPBp.

2. MAIN RESULT

Given a Banach space X , the modulus of convexity $\delta_X(\epsilon)$ of the unit ball B_X is defined by for $0 < \epsilon < 1$,

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x-y\| \geq \epsilon \right\}.$$

A Banach space X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon < 1$. It is well known that every uniformly convex space is reflexive.

In [20], the following result was shown: Let $1 > \epsilon > 0$ be given and X be a reflexive Banach space and Y be a uniformly convex Banach space with modulus of convexity $\delta_X(\epsilon) > 0$. If $T \in S_{\mathcal{L}(X, Y)}$ and $x_1 \in S_X$ satisfy

$$\|Tx_1\| > 1 - \frac{\epsilon}{25} \delta_X\left(\frac{\epsilon}{2}\right),$$

then there exist $S \in S_{\mathcal{L}(X, Y)}$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \epsilon$ and $\|Tx_1 - Sx_2\| < \epsilon$.

This says that for a reflexive space X and a uniformly convex space Y , the pair (X, Y) has a little weaker property than BPBp. The only difference from the BPBp and the above is approximating the image of a point if the given operator almost attains its norm. Since the set of all norm attaining operators is dense in $\mathcal{L}(X, Y)$ for every Y if X is reflexive, the following result generalize the result mentioned above [20].

Proposition 2.1. *Let X be a Banach space and Y be a uniformly convex space. Suppose that the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$. Then, given $0 < \epsilon < 1$, there exists $\eta(\epsilon) > 0$ such that if $T \in S_{\mathcal{L}(X, Y)}$ and $x_1 \in S_X$ satisfy $\|Tx_1\| > 1 - \eta(\epsilon)$, then there exist $S \in S_{\mathcal{L}(X, Y)}$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \epsilon$ and $\|Tx_1 - Sx_2\| < \epsilon$.*

Proof. Let $\delta_Y(\cdot)$ be the modulus of convexity of Y and $0 < \epsilon_1 < \epsilon$. Choose $\epsilon_2 > 0$ such that $(1 - \epsilon_2^2)^3 - 2\epsilon_2 - \epsilon_2^3 > 1 - \delta_Y(\epsilon_1)$ and $\epsilon_2^2 + 2\epsilon_2 + \epsilon_1 < \epsilon$.

We show that $\eta(\epsilon) = \epsilon_2^2$ is a suitable number. Assume $\|Tx_1\| > 1 - \epsilon_2^2$. Choose $y^* \in S_{Y^*}$ such that $y^*Tx_1 = \operatorname{Re} y^*Tx_1 > 1 - \epsilon_2^2$ and define an operator \tilde{T}_1 by

$$\tilde{T}_1x = Tx + \epsilon_2 y^*(Tx)Tx_1 \text{ for every } x \in X.$$

It is easy to see that $1 - \epsilon_2 < (1 - \epsilon_2^2)(1 + \epsilon_2(1 - \epsilon_2^2)) \leq \|\tilde{T}_1x_1\| \leq \|\tilde{T}_1\| \leq 1 + \epsilon_2$.

Let $T_1 = \tilde{T}_1/\|\tilde{T}_1\|$. Since the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$, there exist an operator S and $z \in S_X$ such that $\|T_1 - S\| < \epsilon_2^2$ and $\|Sz\| = \|S\| = 1$. Since $\|Sz - T_1z\| < \epsilon_2^2$, we see that $\|T_1z\| > 1 - \epsilon_2^2$, which means that

$$\|Tz + \epsilon_2 y^*(Tz)Tx_1\| > (1 - \epsilon_2^2)\|\tilde{T}_1\| > (1 - \epsilon_2^2)(1 - \epsilon_2^2)(1 + \epsilon_2(1 - \epsilon_2^2)).$$

Hence, we have $|y^*(Tz)| > (1 - \epsilon_2^2)^3 - 2\epsilon_2 - \epsilon_2^3 > 1 - \delta_Y(\epsilon_1)$. Choose $\alpha = \pm 1$ satisfying $y^*T(\alpha z) = |y^*T(z)|$ and let $x_2 = \alpha z$. Then

$$\left\| \frac{Tx_1 + Tx_2}{2} \right\| \geq \frac{y^*Tx_1 + y^*Tx_2}{2} > 1 - \delta_Y(\epsilon_1).$$

Hence, we see that $\|Tx_1 - Tx_2\| < \epsilon_1$. Moreover,

$$\begin{aligned} \|Sx_2 - Tx_1\| &\leq \|Sx_2 - T_1x_2\| + \|T_1x_2 - \tilde{T}_1x_2\| + \|\tilde{T}_1x_2 - Tx_2\| + \|Tx_2 - Tx_1\| \\ &\leq \|S - T_1\| + \|\tilde{T}_1 - 1\| + \epsilon_2 + \epsilon_1 \\ &< \epsilon_2^2 + \epsilon_2 + \epsilon_2 + \epsilon_1 < \epsilon. \end{aligned}$$

This completes the proof. \square

Now we state the main theorem of this paper.

Theorem 2.2. *Let X be a uniformly convex space and K be a compact Hausdorff space. Then the pair $(C(K), X)$ has the BPBP.*

Before we present the proof of the main result, we begin with preliminary comments on vector measure and two lemmas. Recall that a vector measure $G : \Sigma \rightarrow X$ on a σ -algebra Σ is said to be countably additive if, for every mutually disjoint sequence of Σ -measurable subsets $\{A_i\}_{i=1}^\infty$, we have

$$G\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty G(A_i).$$

For a Σ -measurable subset A , the semi-variation $\|G\|(A)$ of G is defined by

$$\|G\|(A) = \sup\{|x^*G|(A) : x^* \in B_{X^*}\},$$

where $|x^*G|(A)$ is the total variation of the scalar-valued countably additive measure x^*G on A . The vector measure G on a Borel σ -algebra is said to be regular if for each Borel subset E and $\epsilon > 0$ there exists a compact subset K and an open set O such that

$$K \subset E \subset O \quad \text{and} \quad \|G\|(O \setminus K) < \epsilon.$$

It is well known that if X is reflexive, each operator T in $\mathcal{L}(C(K), X)$ has a X -valued countably additive representing Borel measure G and the measure is regular (see [16, VI. Theorem 1, 5 and Corollary 14] for a reference). That is, for all $f \in C(K)$ and $x^* \in X^*$, we have

$$Tf = \int_K f dG, \quad x^*T(f) = \int_K f d^*G \quad \text{and} \quad \|T\| = \|G\|(K).$$

If G is a countably additive representing measure for an operator T in $\mathcal{L}(C(K), X)$, then it is easy to see that for any bounded Borel measurable function $h : K \rightarrow \mathbb{R}$, the mapping S , defined by $Sf = \int fhdG$, is a bounded linear operator and $\|S\| \leq \|T\| \cdot \|h\|_\infty$, where $\|h\|_\infty = \sup\{|h(k)| : k \in K\}$.

Lemma 2.3. *Let G be a countably additive, Borel regular X -valued vector measure on a compact Hausdorff space K with $\|G\|(K) = 1$ and let $0 < \eta, \gamma < 1$. Assume that $f \in S_{C(K)}$ and $x^* \in S_{X^*}$ satisfy*

$$\int_K f dx^*G > 1 - \eta.$$

Then, we have

$$|x^*G|(K \setminus (A_\gamma^+ \cup A_\gamma^-)) < 2\frac{\eta}{\gamma} + \eta,$$

where $A_\gamma^+ = \{t \in K \mid f(t) \geq 1 - \gamma\}$ and $A_\gamma^- = \{t \in K \mid f(t) \leq -1 + \gamma\}$. Moreover, there exist mutually disjoint compact sets F^+, F^- such that x^*G is positive on F^+ , negative on F^- and

$$\int_{(F^+ \cap A_\gamma^+) \cup (F^- \cap A_\gamma^-)} f dx^*G > 1 - 4\frac{\eta}{\gamma}.$$

Proof. The Hahn decomposition of x^*G and the regularity of G show that there exist mutually disjoint compact sets F^+, F^- such that x^*G is positive on F^+ , negative on F^- and $\|G\|(K \setminus (F^+ \cup F^-)) < \eta$.

$$\begin{aligned} 1 - \eta &\leq \int_K f dx^*G = \int_{F^+} f dx^*G + \int_{F^-} f dx^*G + \int_{K \setminus (F^+ \cup F^-)} f dx^*G \\ &= \int_{F^+ \cap A_\gamma^+} f dx^*G + \int_{F^+ \setminus A_\gamma^+} f dx^*G + \int_{F^- \cap A_\gamma^-} f dx^*G + \int_{F^- \setminus A_\gamma^-} f dx^*G + \int_{K \setminus (F^+ \cup F^-)} f dx^*G \\ &\leq x^*G(F^+ \cap A_\gamma^+) + (1 - \gamma)x^*G(F^+ \setminus A_\gamma^+) - x^*G(F^- \cap A_\gamma^-) - (1 - \gamma)x^*G(F^- \setminus A_\gamma^-) + \eta \\ &= x^*G(F^+) - x^*G(F^-) - \gamma(x^*G(F^+ \setminus A_\gamma^+) - x^*G(F^- \setminus A_\gamma^-)) + \eta. \end{aligned}$$

Since $x^*G(F^+) - x^*G(F^-) = |x^*G|(F^+ \cup F^-) \leq \|G\|(K) = 1$, we get

$$|x^*G|((F^+ \setminus A_\gamma^+) \cup (F^- \setminus A_\gamma^-)) = x^*G(F^+ \setminus A_\gamma^+) - x^*G(F^- \setminus A_\gamma^-) \leq 2\frac{\eta}{\gamma}.$$

This shows that

$$\begin{aligned} |x^*G|(K \setminus (A_\gamma^+ \cup A_\gamma^-)) &\leq |x^*G|(K \setminus (F^+ \cup F^-)) + |x^*G|(F^+ \cup F^- \setminus (A_\gamma^+ \cup A_\gamma^-)) \\ &\leq \|G\|(K \setminus (F^+ \cup F^-)) + |x^*G|((F^+ \setminus A_\gamma^+) \cup (F^- \setminus A_\gamma^-)) \\ &< 2\frac{\eta}{\gamma} + \eta \end{aligned}$$

and

$$\begin{aligned} \int_{(F^+ \cap A_\gamma^+) \cup (F^- \cap A_\gamma^-)} f dx^*G &= \int_{F^+ \cup F^-} f dx^*G - \int_{(F^+ \setminus A_\gamma^+) \cup (F^- \setminus A_\gamma^-)} f dx^*G \\ &\geq \int_K f dx^*G - \|G\|(K \setminus (F^+ \cup F^-)) - |x^*G|((F^+ \setminus A_\gamma^+) \cup (F^- \setminus A_\gamma^-)) \\ &> 1 - 2\eta - 2\frac{\eta}{\gamma} > 1 - 4\frac{\eta}{\gamma}. \end{aligned}$$

This completes the proof. \square

Lemma 2.4. *Let X be a uniformly convex space with the modulus of convexity δ_X and $T \in S_{\mathcal{L}(C(K), X)}$ be an operator represented by the countably additive, Borel regular vector measure G . Let $0 < \epsilon < 1$ and A be a Borel set of K . Suppose that an operator S , defined by $Sf = \int_A fdG$, satisfies $\|S\| > 1 - \delta_X(\epsilon)$. Then*

$$\|T - S\| = \sup_{f \in B_{C(K)}} \left\| \int_{K \setminus A} fdG \right\| < \epsilon.$$

Proof. Choose $x^* \in S_{X^*}$, $f_0 \in S_{C(K)}$ such that $\|Sf_0\| = x^*Sf_0 > 1 - \delta_X(\varepsilon)$. By the regularity of G , we may choose a compact set $A_1 \subset A$ such that

$$\int_{A_1} f_0 dx^* G > 1 - \delta_X(\varepsilon).$$

Fix a closed set $B \subset K \setminus A$ and $g \in B_{C(B)}$. Then, choose $g_+, g_- \in B_{C(K)}$ satisfying

$$\begin{aligned} g_+(t) &= g_-(t) = f_0(t) \quad \text{for } t \in A_1 \quad \text{and} \\ g_+(t) &= -g_-(t) = g(t) \quad \text{for } t \in B. \end{aligned}$$

So, we have

$$1 - \delta_X(\varepsilon) < \int_{A_1} f_0 dx^* G \leq \left\| \int_{A_1} f_0 dG \right\| = \frac{1}{2} \left\| \int_{A_1 \cup B} g_+ dG + \int_{A_1 \cup B} g_- dG \right\|.$$

Note that $\left\| \int_{A_1 \cup B} g_+ dG \right\|, \left\| \int_{A_1 \cup B} g_- dG \right\| \leq 1$. Thus, from the uniform convexity of X , we get that

$$\left\| 2 \int_B g dG \right\| = \left\| \int_{A_1 \cup B} g_+ dG - \int_{A_1 \cup B} g_- dG \right\| < \varepsilon.$$

This implies $\|T - S\| < \varepsilon$ and the proof is done. \square

Proof of Theorem 2.2. Let δ_X be the modulus of convexity for B_X . Fix $0 < \varepsilon < \frac{1}{2^8}$ and let η be the function which appears in Proposition 2.1 for the pair $(C(K), X)$, and let $\gamma(t) = \min \left\{ \eta(t), \delta_X(t), \frac{t}{3} \right\}$ for $t \in (0, 1)$. Assume that $T \in S_{\mathcal{L}(C(K), X)}$ and $f_0 \in S_{C(K)}$ satisfy that

$$\|Tf_0\| > 1 - \frac{\varepsilon}{8} \gamma \left(\frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right) \right).$$

Let G be the representing vector measure for T which is countably additive Borel regular on K . Choose $x_1^* \in S_{X^*}$ such that $x_1^*Tf_0 > 1 - \frac{\varepsilon}{8} \gamma \left(\frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right) \right)$. By Lemma 2.3 there exist two mutually disjoint compact sets F^+, F^- such that x^*G is positive on F^+ , negative on F^- and

$$\int_{(F^+ \cap A_{\varepsilon/2}^+) \cup (F^- \cap A_{\varepsilon/2}^-)} f dx^* G > 1 - \gamma \left(\frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right) \right),$$

where $A_{\varepsilon/2}^+ = \{t \in K \mid f_0(t) \geq 1 - \frac{\varepsilon}{2}\}$ and $A_{\varepsilon/2}^- = \{t \in K \mid f_0(t) \leq -1 + \frac{\varepsilon}{2}\}$.

Let $A_1 = F^+ \cap A_{\varepsilon/2}^+$, $A_2 = F^- \cap A_{\varepsilon/2}^-$ and $A = A_1 \cup A_2$. Then, define $S_1 \in B_{\mathcal{L}(C(K), X)}$ by $S_1 f = \int_A f dG$ for every $f \in C(K)$. Then Lemma 2.4 shows that $\|T - S_1\| < \frac{\varepsilon}{6}$. Choose $f_1 \in S_{C(K)}$ such that

$$\begin{aligned} f_1(t) &= 1 \quad \text{for } t \in A_1 \quad \text{and} \\ f_1(t) &= -1 \quad \text{for } t \in A_2. \end{aligned}$$

For $f \in C(K)$, the restriction of f to A will be denoted by $f|_A$. Now consider S_1 as an operator in $\mathcal{L}(C(A), X)$. Then we have

$$\|S_1(f_1|_A)\| > 1 - \gamma \left(\frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right) \right),$$

So Proposition 2.1 shows that there exist $S_2 \in \mathcal{L}(C(A), X)$ and $f_2 \in S_{C(A)}$ such that $\|S_2 f_2\| = 1$, $\left\| S_2 - \frac{S_1}{\|S_1\|} \right\| < \frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right)$ and $\left\| S_2 f_2 - \frac{S_1(f_1|_A)}{\|S_1\|} \right\| < \frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right)$. Let G' be the representing vector measure for S_2 which is countably additive Borel regular on A . Choose $x_2^* \in S_{X^*}$ so that $x_2^* S_2 f_2 = \|S_2 f_2\| = \int_A f_2 dx_2^* G' = 1$.

Since

$$\begin{aligned} x_2^* S_2(f_1|_A + f_2) &\geq 2x_2^* S_2 f_2 - \|S_2 f_2 - S_2(f_1|_A)\| \\ &\geq 2 - \left\| S_2 f_2 - \frac{S_1(f_1|_A)}{\|S_1\|} \right\| - \left\| \frac{S_1(f_1|_A)}{\|S_1\|} - S_2(f_1|_A) \right\| \\ &> 2 \left(1 - \frac{\varepsilon}{6} \delta_X \left(\frac{\varepsilon}{6} \right) \right), \end{aligned}$$

we get

$$\int_A \frac{f_1 + f_2}{2} dx^* G' > 1 - \frac{\epsilon}{6} \delta_X \left(\frac{\epsilon}{6} \right).$$

By applying Lemma 2.3 again, we get a compact subset F of A such that

$$F \subset \{t \in A : |f_1(t) + f_2(t)| > 2(1 - \epsilon)\}$$

and

$$\left\| \int_F \frac{f_1 + f_2}{2} dG' \right\| > 1 - \delta_X \left(\frac{\epsilon}{6} \right).$$

Let $B = \{t \in A : f_1(t)f_2(t) \geq 0\}$. Then, $F \subset B$ and

$$\sup_{f \in B_{C(A)}} \left\| \int_B f dG' \right\| \geq \left\| \int_F \frac{f_1 + f_2}{2} dG' \right\| > 1 - \delta_X \left(\frac{\epsilon}{6} \right).$$

By Lemma 2.4, we have

$$\sup_{f \in B_{C(K)}} \left\| \int_{A \setminus B} f dG' \right\| < \frac{\epsilon}{6}.$$

Define $S \in \mathcal{L}(C(A), X)$ by, for $f \in C(A)$,

$$Sf = \int_B f dG' - \int_{A \setminus B} f dG'$$

and let

$$f_3 = \begin{cases} |f_2| & \text{for } t \in A_1, \\ -|f_2| & \text{for } t \in A_2. \end{cases}$$

So $f_3 \in C(A)$ and $f_3 = f_2 \chi_B - f_2 \chi_{A \setminus B}$, where χ_S is the characteristic function on a set S . Hence we have $Sf_3 = S_2 f_2$, $\|Sf_3\| = \|S\| = 1$ and $\|S - S_2\| < \frac{\epsilon}{3}$. On the other hand, we have $\|2f_3 - f_1|_A\| \leq 1$. Since X is uniformly convex and we have $Sf_3 = \frac{S(f_1|_A) + S(2f_3 - f_1|_A)}{2}$, we get

$$Sf_3 = S(f_1|_A) = S(2f_3 - f_1|_A).$$

We now consider S_1, S_2, S as operators in $\mathcal{L}(C(K), X)$ using the canonical extension. That is, $S(f) = S(f|_A)$, $S_i(f) = S_i(f|_A)$ for all $f \in C(K)$ and for $i = 1, 2$. Let C be the compact subset defined by

$$C = \{t \in K : |f_1(t) - f_0(t)| \geq \epsilon\}.$$

Note that A and C are mutually disjoint. Indeed, if $t \in A$, then $|f_0(t) - f_1(t)| \leq \epsilon/2$. So there is $\phi \in C(K)$ such that $0 \leq \phi \leq 1$, $\phi(k) = 1$ for $k \in A$ and $\phi(k) = 0$ for $k \in C$. Let $g = \phi f_1 + (1 - \phi)f_0$. Then we see that $\|Sg\| = 1$,

$$\begin{aligned} \|S - T\| &\leq \|S - S_2\| + \|S_2 - \frac{S_1}{\|S_1\|}\| + \|\frac{S_1}{\|S_1\|} - S_1\| + \|S_1 - T\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon \end{aligned}$$

and $\|g - f_0\| = \sup_{k \in K \setminus C} |\phi(k)(f_1(k) - f_0(k))| < \epsilon$. This completes the proof. \square

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